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ORDERINGS AND RISK PROBABILITY FUNCTIONALS IN PORTFOLIO THEORY

BY

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Abstract. This paper studies and describes stochastic orderings of risk/reward positions in order to define in a natural way risk/reward measures consistent/isotonic to investors' preferences. We begin by discussing the connection between the theory of probability metrics, risk measures, distributional moments, and stochastic orderings. Then we examine several classes of orderings which are generated by risk probability functionals. Finally, we demonstrate how further orderings could better specify the investor's attitude toward risk.

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1. INTRODUCTION

In this paper, we describe the admissible classes of probability functionals that are consistent with a given order of preferences. To classify the orderings and risk probability functionals, we distinguish between primary and compound probability functionals; between uncertainty and risk orderings/measures; between orderings and survival/dual orderings; and between bounded and unbounded orderings. By doing so, we present a general and unifying framework to understand the connections between the investor's preferences that are consistent to a given order and choice problem.

We first discuss the links between continua stochastic dominance orders, dual stochastic dominance rules based on Lorenz orders, and the different distributional moments of a portfolio of assets returns; see Fishburn [6], [7] and Muliere and Scarsini [19]. We tie together the consistency-isotonicity of risk and reward measures with the classical orderings. We study the properties that a probability functional must satisfy to solve optimal choice problems. The theory of probability functionals and metrics was developed by Zolotarev and his students to solve stability problems, see Rachev [27] and the references therein. Furthermore, there exists a strong connection between probability functionals and orderings; see, among others, Kakosyan et al. [13], Kalashnikov and Rachev [14], and Rachev and Rüschendorf [29], [30].

In this paper, we discuss the static approach to the theory of choice under risk and uncertainty. In particular, we are interested in the economic use of probability functionals to optimize choices for a given order of investors' preferences. As a consequence of this discussion, we propose a new set of orderings, risk and reward measures that are coherent to investors' choices. The new probability functionals and orderings generalize those found in the literature and are strictly related to the theory of choice under uncertainty (see, among others, von Neumann and Morgenstern [37], Machina [17], Yaari [38], Gilboa and Schmeidler [11], and Maccheroni et al. [16]) and to the theory of probability functionals and metrics; see Rachev [27] and the references therein. While the new orderings serve to further characterize and specify the investors' choices/preferences, the new risk measures should be used either to minimize the risk of a portfolio of financial assets or to minimize its distance from a given benchmark, see Rachev et al. [28], Stoyanov et al. [33], and Ortobelli et al. [25]. We will call these new measures/orderings "FORS measures/orderings". We show how one can generate further orderings and measures by using the Mellin transform when applied to the fractional integral.

In the next section, we examine continua and dual stochastic dominance and their connection to the distributional moments of portfolios. In Section 3, we describe how to use probability functionals to define new orderings and portfolio risk measures. Finally, we briefly summarize the results.

2. CONTINUA AND DUAL STOCHASTIC DOMINANCE THEORY

In this section, we study the stochastic orders in a complete probability space $(\Omega, \Im, P)$. By doing so, we take the perspective of an investor who wants to solve a portfolio selection problem. In particular, we denote by $L^0(\Im)$ the space of all real-valued random variables defined on $(\Omega, \Im, P)$ while

$$L^p(\Im) = \{X : (\Omega, \Im, P) \to R \mid E(|X|^p) < +\infty\}.$$

Recall that X dominates Y with respect to n (integer) order stochastic dominance ($X \underset{n}{\geqslant} Y$) if and only if $E\big(u(X)\big) \geqslant E\big(u(Y)\big)$ for every utility function u whose derivatives satisfy the inequalities $(-1)^{k+1}u^{(k)} \geqslant 0$ for $k = 1, \ldots, n$, if and only if for every real t we have

$$F_X^{(n)}(t) := \int_{-\infty}^{t} F_X^{(n-1)}(u)du \leqslant F_Y^{(n)}(t).$$

Furthermore, we observe that for any $m > n$, $X \underset{n}{\geqslant} Y$ implies $X \underset{m}{\geqslant} Y$. In addition, we state that X dominates Y in the sense of Rothschild and Stiglitz [31]

($X\ R-S\ Y$) if and only if $E\big(u(X)\big)\geqslant E\big(u(Y)\big)$ for every concave utility function u, if and only if $X\underset{2}{\geqslant}Y$ and $E(X)=E(Y)$. This order is also called *concave* in the ordering literature; see, among others, Shaked and Shanthikumar [32] and Müller and Stoyan [21]. Moreover, all these relations can be easily generalized in continuous terms.

2.1. Continua, survival, bounded/unbounded stochastic dominance rules. Fishburn [6], [7] considers continuous orders applied to bounded and unbounded random variables. In the following, we further characterize and generalize these orders. This extension is possible because the first lemma found in Fishburn [6] is still valid following the Fubini–Tonelli theorem; see also Miller and Ross [18] and Zhang and Jin [39].

LEMMA 2.1. *Let (R,B_R,μ) be the real space with the Borel sigma algebra B_R and a positive sigma finite measure μ. Then for any $-\infty\leqslant a<z\leqslant+\infty$*

$$\int_a^z(z-x)^{v-1}\Big(\int_a^{x^-}(x-y)^{\alpha-1}d\mu(y)\Big)dx=B\,(\alpha,v)\int_a^{z^-}(z-y)^{\alpha+v-1}d\mu\,(y),$$

$$\int_a^z(x-a)^{v-1}\Big(\int_{x^+}^{z}(y-x)^{\alpha-1}d\mu\,(y)\Big)dx=B\,(\alpha,v)\int_{a^+}^{z}(y-a)^{\alpha+v-1}d\mu\,(y),$$

where

$$B\,(\alpha,v)=\frac{\Gamma(\alpha)\Gamma(v)}{\Gamma(\alpha+v)}\quad \textit{and}\quad \Gamma(t)=\int_0^{+\infty}z^{t-1}e^{-z}dz.$$

These relations are still valid if α,v are complex numbers with $\operatorname{Re}\alpha,\operatorname{Re}v>0$ and $B(\alpha,v)$ is the beta function with complex arguments.

We assume that $F_X^{(1)}=F_X$ and put $a=\inf\{x\mid F_X(x)>0\}$. Using the definition of fractional integral (see Erdelyi and McBride [5] and Miller and Ross [18]), we obtain for every real $\alpha>0$ and $\alpha\neq 1$, $F_X^{(\alpha)}(t)=0$, for all $t\leqslant a$ and for every $t>a$

$$F_X^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha)}\int_a^{t^-}(t-y)^{\alpha-1}dF_X(y)=\frac{E\big((t-X)_{++}^{\alpha-1}\big)}{\Gamma(\alpha)},\tag{2.1}$$

where $(t-x)_{++}^{\alpha-1}=(t-x)^{\alpha-1}I_{[x<t]}$ and $I_{[x<t]}$ is the indicator function equal to 1 if $x<t$, and 0 otherwise. Thus, $F_X^{(\alpha)}$ is a positive continuous function for $\alpha>1$; it is right continuous for $\alpha=1$ and left continuous for $\alpha\in(0,1)$. A slightly different definition (see Fishburn [7]) is necessary for $\alpha\in(0,1)$ in order to include the probability measures that satisfy $P(X=t_i)>0$ for some real numbers t_i. Analogously, we can use the survival function $\bar{F}_X^{(1)}(x)=P(X>x)=1-F_X(x)$

and we obtain, for every positive real $\alpha \neq 1$ and for every $t < b$,

$$\bar{F}_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (y-t)^{\alpha-1} dF_X(y) = \frac{E\big((X-t)_+^{\alpha-1}\big)}{\Gamma(\alpha)}, \tag{2.2}$$

and $\bar{F}_X^{(\alpha)}(t) = 0$ for all $t \geqslant b$, where $b = \sup\{x \mid F_X(x) < 1\}$. In this case, the function $\bar{F}_X^{(\alpha)}(x)$ is right continuous for all $\alpha \in (0, 1]$ and it is continuous for $\alpha > 1$. In particular, when X is a continuous random variable, $\bar{F}_X^{(\alpha)}(u) = F_{-X}^{(\alpha)}(-u)$ for every $u \in [a, b]$ and $\alpha > 0$. From Lemma 2.1 it follows that, if μ is the probability measure obtained by the right continuous distribution function of X, $F_X^{(1)}(y) = F_X(y) = \mu(y)$ or, by the survival function $\mu(y) = \bar{F}_X^{(1)}(y)$, we obtain

$$F_X^{(\alpha)}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha-v)} \displaystyle\int_a^{t^-} (t-u)^{\alpha-v-1} F_X^{(v)}(u)du \\ \qquad \text{for all } \alpha > v \geqslant 1 \text{ or for all } 1 > \alpha > v > 0, \\ \lim\limits_{t_n \searrow t} \dfrac{1}{\Gamma(\alpha-v)} \displaystyle\int_a^{t_n} (t_n-u)^{\alpha-v-1}\, F_X^{(v)}(u)du \\ \qquad \text{for all } \alpha \geqslant 1 > v > 0, \end{cases} \tag{2.3}$$

$$\bar{F}_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha-v)} \int_t^b (u-t)^{\alpha-v-1} \bar{F}_X^{(v)}(u)du \quad \text{for all } \alpha > v > 0. \tag{2.4}$$

We can define stochastic orders as follows:

DEFINITION 2.1. For every $\alpha > 0$, we state that X *dominates* Y *with respect to the* α *bounded stochastic dominance order* ($X \underset{\alpha}{\overset{b}{\geqslant}} Y$) iff $F_X^{(\alpha)}(t) \leqslant F_Y^{(\alpha)}(t)$ for every t belonging to $\operatorname{supp}\{X, Y\} \equiv [a, b]$, where $a = \inf\{x \mid F_X(x) + F_Y(x) > 0\}$, and $b = \{x \mid F_X(x) + F_Y(x) < 2\}$. We state that X *strictly dominates* Y *with respect to the* α *bounded order* (namely, $X \underset{\alpha}{\overset{b}{>}} Y$) iff $X \underset{\alpha}{\overset{b}{\geqslant}} Y$ and $F_X \neq F_Y$.

Moreover, following Fishburn [7]: for every $\alpha \geqslant 1$, X dominates Y with respect to the α stochastic dominance order ($X \underset{\alpha}{\geqslant} Y$) iff $F_X^{(\alpha)}(t) \leqslant F_Y^{(\alpha)}(t)$ for every real t. X strictly dominates Y with respect to the α order (namely, $X \underset{\alpha}{>} Y$) iff $X \underset{\alpha}{\geqslant} Y$ and $F_X \neq F_Y$. Since $X \underset{\alpha}{\geqslant} Y$ ($X \underset{\alpha}{\overset{b}{\geqslant}} Y$) is simply $X \underset{\alpha}{>} Y$ ($X \underset{\alpha}{\overset{b}{>}} Y$) plus the identity relation, we shall consider only $\underset{\alpha}{>}$ ($\underset{\alpha}{\overset{b}{>}}$) explicitly. As proven by Fishburn [7], bounded and unbounded orderings $\underset{\alpha}{\overset{b}{>}}$, $\underset{\alpha}{>}$ are equivalent among random variables with finite expected values if and only if $\alpha \in [1, 2]$. When $\alpha \notin [1, 2]$, the orders $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$ do not generally coincide. Similarly we define the *survival bounded order* ($X \overset{a}{\underset{\mathrm{sur}\,\alpha}{\geqslant}} Y$) iff $\bar{F}_X^{(\alpha)}(t) \leqslant \bar{F}_Y^{(\alpha)}(t)$ for every t that belongs to $\operatorname{supp}\{X, Y\} \equiv [a, b]$ (and the *survival unbounded order*: $X \underset{\mathrm{sur}\,\alpha}{\geqslant} Y$ iff $\bar{F}_X^{(\alpha)}(t) \leqslant \bar{F}_Y^{(\alpha)}(t)$ for every real t). We prefer to concentrate on stochastic dominance orders because when $\alpha > 1$, we have

$$\bar{F}_X^{(\alpha)}(u) = E\big((X-u)_+^{\alpha-1}\big)/\Gamma(\alpha) = F_{-X}^{(\alpha)}(-u)$$

and the results are equivalent to those obtained for orders on the opposite of random variables.

Even if Definition 2.1 generalizes the orders proposed by Fishburn [6], [7] to α bounded orders with $\alpha \in (0,1)$ that imply first stochastic dominance, in many cases we cannot compare random variables with respect to these orders. In particular, if $X \overset{b}{\underset{\alpha}{>}} Y$ with $\alpha \in (0,1)$, a point $t < \sup\{x \mid F_X(x) + F_Y(x) < 2\}$ such that $0 < P(Y = t) < P(X = t)$ cannot exist because in the right neighborhood of t we have $F_X^{(\alpha)}(t^+) > F_Y^{(\alpha)}(t^+)$. In addition, as follows from the proposition below, we cannot express the α order $X \underset{\alpha}{>} Y$ for any $\alpha \in (0,1)$.

PROPOSITION 2.1. *For any pair of bounded* (*from above or/and from below*) *random variables X and Y that are continuous on the extremes of their support*, *there is no $\alpha \in (0,1)$ such that $F_X^{(\alpha)}(t) \leqslant F_Y^{(\alpha)}(t)$ for all $t \in \operatorname{supp}(X,Y)$. In addition*, *for any pair of random variables X and Y*, *there is no $\alpha \in (0,1)$ such that $F_X^{(\alpha)}(t) \leqslant F_Y^{(\alpha)}(t)$ for every real t.*

Proof. Consider $\alpha \in (0,1)$. Let X and Y be bounded random variables continuous on the extremes of the support. Then

$$X, Y < \sup\{x \mid F_X(x) + F_Y(x) < 2\} = b < +\infty.$$

Suppose $X \overset{b}{\underset{1}{>}} Y$. Then we obtain

$$\Gamma(\alpha)F_X^{(\alpha)}(b) = \int_{-\infty}^{b} (b-x)^{\alpha-1}\,dF_X(x) \geqslant \int_{-\infty}^{b} (b-x)^{\alpha-1}\,dF_Y(x) = F_Y^{(\alpha)}(b)\Gamma(\alpha)$$

because X and Y are continuous on the extreme b and the function $u(x) = (b-x)^{\alpha-1}I_{[x<b]}$ is an increasing function for $x \leqslant b$ and

$$\Gamma(\alpha)F_Y^{(\alpha)}(b) = E\big(u(Y)\big), \quad \Gamma(\alpha)F_X^{(\alpha)}(b) = E\big(u(X)\big).$$

Furthermore, because the function $g(\alpha) = F_X^{(\alpha)}(b) = E\left((b-X)^{\alpha-1}\right)$ is analytic, there exists $r \in (\alpha, 1)$ such that $F_X^{(r)}(b) > F_Y^{(r)}(b)$; otherwise $F_X = F_Y$ because using the Mellin transform we can univocally determine the distribution functions. In the case where X and Y are bounded random variables not necessarily continuous on the extremes $X, Y < b < +\infty$, for every increasing function $v(x) = (t-x)^{\alpha-1} I_{[x \leqslant b]}$ and $t > b$ it follows that

$$E\big(v(X)\big) = \int_{-\infty}^{b} (t-x)^{\alpha-1}\, dF_X(x) \geqslant \int_{-\infty}^{b} (t-x)^{\alpha-1}\, dF_Y(x) = E\big(v(Y)\big),$$

and similarly the inequality is strict for some t. There are analogous considerations when $X, Y > \inf\{x \mid F_X(x) + F_Y(x) > 0\} = a > -\infty$ because $-X, -Y < -a$ and $X \overset{b}{\underset{1}{>}} Y$ iff $-Y \overset{-a}{\underset{1}{>}} -X$. Next, suppose X and Y are a pair of random variables such that $X \overset{b}{\underset{1}{>}} Y$. Observe that $X \overset{b}{\underset{1}{>}} Y$ iff $X_-^{(M)} \overset{M}{\underset{1}{\geqslant}} Y_-^{(M)}$, $X_+^{(M)} \overset{b}{\underset{1}{\geqslant}} Y_+^{(M)}$ and at least one dominance is strict, where $X = X_+^{(M)} + X_-^{(M)}$ and $X_+^{(M)} = XI_{[X \geqslant M]}$, $X_-^{(M)} = XI_{[X<M]}$ for every $M \in \operatorname{supp}\{X, Y\}$. Thus the assertion follows. ∎

Due to this proposition, we cannot compare random variables according to the α bounded order with $\alpha \in (0,1)$ except in a few cases. However, although α bounded orders with $\alpha \in (0,1)$ are not applicable in many cases, they could serve to rank financial losses and truncated variables. This is why this generalization could be interesting from a financial point of view. Typically, for every pair of random variables X and Y, with density of probability such that $f_X(t) \leqslant f_Y(t)$ for all $t < M$ and $P(X \leqslant M) = P(Y \leqslant M) = 1$, we have $F_X^{(\alpha)}(t) \leqslant F_Y^{(\alpha)}(t)$ for all $t \in \operatorname{supp}\{X, Y\}$, and $X \overset{M}{\underset{\alpha}{>}} Y$ for every $\alpha > 0$. The following example shows the use of the α order for truncated variables.

EXAMPLE 2.1. Let Y_1 and Y_2 be two financial losses with truncated Gaussian distribution functions:

(a) the function

$$F_{Y_1}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2\right) dx$$

for $t < \mu < +\infty$ is equal to a Gaussian $N(\mu, \sigma)$ and $F_{Y_1}(t) = 1$ for $t \geqslant \mu$;

(b) the function

$$F_{Y_2}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\left(\frac{x-\mu-\varepsilon}{\sqrt{2}\sigma}\right)^2\right) dx$$

for $t < \mu < +\infty$ is equal to a Gaussian $N(\mu + \varepsilon, \sigma)$ with $\varepsilon > 0$ and $F_{Y_1}(t) = 1$ for $t \geqslant \mu$.

Since the probability density of the two losses are $f_{Y_2}(t) \leqslant f_{Y_1}(t)$ for all $t < \mu$ and $P(Y_1 \leqslant \mu) = P(Y_2 \leqslant \mu) = 1$, $F_{Y_2}^{(\alpha)}(t) \leqslant F_{Y_1}^{(\alpha)}(t)$ for all $t \in \operatorname{supp}\{Y_1, Y_2\}$, and $Y_2 \overset{\mu}{\underset{\alpha}{>}} Y_1$ for every $\alpha > 0$. Therefore, all investors would prefer loss Y_1 to Y_2.

In addition, equation (2.3) extends the use of functions $F_X^{(\alpha)}$. From a practical point of view, this has an immediate effect as shown in the following:

REMARK 2.1. *The following implications hold*:

(1) $X \underset{\alpha}{>} Y$ *implies* $X \overset{b}{\underset{\alpha}{>}} Y$. *These orders coincide if and only if* $\alpha \in [1, 2]$. *Therefore, every outcome of the* $\overset{b}{\underset{\alpha}{>}}$ *order is true when* $\underset{\alpha}{>}$ *holds, but the converse is not generally true if* $\alpha \notin [1, 2]$.

(2) *For every* $\alpha > v > 0$, $F_X^{(v)}(t) \leqslant F_Y^{(v)}(t)$ *for all* $t \in \operatorname{supp}\{X, Y\}$ *implies* $F_X^{(\alpha)}(t) \leqslant F_Y^{(\alpha)}(t)$ *for all* $t \in \operatorname{supp}\{X, Y\}$. *In particular, the order* $X \overset{b}{\underset{v}{>}} Y$ $(X \underset{v}{>} Y$ *if* $v \geqslant 1)$ *implies the order* $X \overset{b}{\underset{\alpha}{>}} Y$ $(X \underset{\alpha}{>} Y)$.

(3) $X \overset{b}{\underset{1}{>}} Y$ *if and only if* $X_-^{(M)} \overset{M}{\underset{1}{\geqslant}} Y_-^{(M)}$, $X_+^{(M)} \overset{b}{\underset{1}{\geqslant}} Y_+^{(M)}$ *and at least one dominance is strict, where* $X = X_+^{(M)} + X_-^{(M)}$ *and* $X_+^{(M)} = XI_{[X \geqslant M]}$; $X_-^{(M)} = XI_{[X<M]}$ *for every* $M \in \operatorname{supp}\{X, Y\}$. *In addition*, $X \overset{b}{\underset{\alpha}{\geqslant}} Y$ *with* $\alpha > 1$ *implies* $X_-^{(M)} \overset{M}{\underset{\alpha}{\geqslant}} Y_-^{(M)}$ *for any given* $M \in \operatorname{supp}\{X, Y\}$.

(4) $X \underset{\alpha}{\geqslant} Y$ $(X \overset{b}{\underset{\alpha}{\geqslant}} Y)$ *if and only if* $cX + t \underset{\alpha}{\geqslant} cY + t$ $(cX + t \overset{cb+t}{\underset{\alpha}{\geqslant}} cY + t)$ *for every* $t \in R$, $c > 0$, $\alpha > 0$.

P r o o f. Points (1) and (2) are a logical consequence of the previous discussion and of formula (2.3). Point (4) is proved by the equality

$$F_{cX+t}(x) = F_X\left(\frac{x-t}{c}\right) \qquad \text{for every } t, x \in R, c > 0.$$

Thus, with this affine transformation, the assertion follows for any $\alpha > 0$. Point (3) follows from the definition of the distribution functions $F_{X_+^{(M)}}(t) = 0$ for all $t < M$ and $F_{X_+^{(M)}}(t) = F_X(t)$ for all $t \geqslant M$, while $F_{X_-^{(M)}}(t) = F_X(t)$ for all $t < M$ and $F_{X_-^{(M)}}(t) = 1$ for all $t \geqslant M$. Thus, for every $M \in R$, the order $X \overset{b}{\underset{\alpha}{\geqslant}} Y$ implies $X_-^{(M)} \overset{M}{\underset{\alpha}{\geqslant}} Y_-^{(M)}$. Observe that generally we cannot say that $X \underset{\alpha}{\geqslant} Y$ implies $X_-^{(M)} \underset{\alpha}{\geqslant} Y_-^{(M)}$. ∎

There is a strong connection between moments and stochastic orders as many authors have pointed out (see, among others, Fishburn [8] and O'Brien [22]). Proposition 2.2 summarizes some of these results and provides necessary conditions on moments for α stochastic orders. It is interesting to observe that generally these implications do not always hold when we consider α bounded stochastic orders.

PROPOSITION 2.2. *The following implications hold*:

(a) *Suppose* $X \underset{\alpha}{>} Y$ *and the moments of* X *and* Y *through integer* n *are finite for* $n-1<\alpha \leqslant n$. *Then*

$$\big(E(X),\ldots,E(X^n)\big) \neq \big(E(Y),\ldots,E(Y^n)\big)$$

and $(-1)^{k+1}E(X^k) > (-1)^{k+1}E(Y^k)$ *for minimum integer* k *such that* $E(X^k) \neq E(Y^k)$.

(b) *If* $X \overset{b}{\underset{\alpha}{\geqslant}} Y$ *with* $\alpha>1$ *implies*

$$\begin{aligned} \frac{1}{2}E(|X_1-X_2|^{\alpha-1}) &\leqslant E\big((X_1-Y_1)_+^{\alpha-1}\big), \\ E\big((Y_1-X_1)_+^{\alpha-1}\big) &\leqslant \frac{1}{2}E(|Y_1-Y_2|^{\alpha-1}), \end{aligned} \tag{2.5}$$

where Y_1, Y_2 *are independent realizations of* Y, X_1, X_2 *are independent realizations of* X, *and* X_1, Y_1 *are independent.*

Proof. The first part of the proposition summarizes one of Fishburn's [8] and O'Brien's [22] results. In order to prove the inequalities given by (2.5), recall that if $X \overset{b}{\underset{\alpha}{\geqslant}} Y$, then $F_{X_2}^{(\alpha)}(X_1) \leqslant F_{Y_1}^{(\alpha)}(X_1)$; $F_{X_1}^{(\alpha)}(Y_1) \leqslant F_{Y_2}^{(\alpha)}(Y_1)$. If we apply the Fubini theorem to the expected value of these random variables, we get for all $\alpha \geqslant 1$:

$$\begin{aligned} \Gamma(\alpha)E\big(F_{X_2}^{(\alpha)}(X_1)\big) &= \frac{1}{2}E(|X_1-X_2|^{\alpha-1}) \\ &\leqslant \Gamma(\alpha)E\big(F_{Y_1}^{(\alpha)}(X_1)\big) = E\big((X_1-Y_1)_+^{\alpha-1}\big). \end{aligned}$$

Similarly we obtain the other inequality. ∎

As for integer orders, we can characterize stochastic orders with respect to a given class of utility functions. In particular, as the result of the previous lemma and of Fishburn [6], [7], we observe that $\underset{\alpha}{\geqslant}$ is a reflexive and transitive preorder, while $\underset{\alpha}{>}$ is a strict partial order (asymmetric and transitive) on the space

$$\tilde{L}^{\alpha-1} = \begin{cases} \{X \mid E(|X|^{\alpha-1}) < +\infty\} & \text{if } \alpha \neq 1,\ \alpha>0, \\ \text{all r.v. } X & \text{if } \alpha = 1. \end{cases}$$

Moreover, for every pair of random variables $X, Y \in \tilde{L}^{\alpha-1}$, $X \underset{\alpha}{\geqslant} Y$ if and only if $E\big(u(X)\big) \geqslant E\big(u(Y)\big)$ for all utility functions u belonging to

$$U_\alpha = \Big\{u(x) = c - \int_{x^+}^{+\infty} (y-x)^{\alpha-1} dv(y) \mid c, x \in R;$$
$$\text{where } v \text{ is a positive } \sigma\text{-finite measure: } \int_{-\infty}^{+\infty} |y|^{\alpha-1} dv(y) < \infty\Big\}.$$

In particular, for every random variable $X \in \tilde{L}^{\alpha-1}$, all utility functions

$$-\Gamma(\alpha)\bar{F}_X^{(\alpha)}(x) = -\int_{x^+}^{+\infty} (t-x)^{\alpha-1}\, dF_X(t) = -E\big((X-x)_+^{\alpha-1}\big)$$

belong to U_α. Similarly, for every pair of random variables $X, Y \in \tilde{L}^{\alpha-1}$ with support on $[a, b]$ $(a, b \in \bar{R})$, $X \overset{b}{\underset{\alpha}{\geqslant}} Y$ if and only if $E\big(u(X)\big) \geqslant E\big(u(Y)\big)$ for all utility functions u belonging to

$$U_\alpha^b = \Big\{u : [a, b] \to R \mid u(x) = c - \int_{x^+}^{b} (y-x)^{\alpha-1} dv(y) - k(b-x)^{\alpha-1};$$
$$c \in R, k \geqslant 0; \text{where } v \text{ is a positive } \sigma\text{-finite measure: } \int_a^b |y|^{\alpha-1}\, dv(y) < \infty\Big\}.$$

The classes U_α and U_α^b are closed under positive affine transformations and are sufficient to characterize the α stochastic order $(\underset{\alpha}{\geqslant}, \overset{b}{\underset{\alpha}{\geqslant}})$, although more general base classes could be used. On the other hand, Fishburn [6], [7] and Müller [20] prove that $U_\alpha \supseteq U_\beta$ $(U_\alpha^b \supseteq U_\beta^b)$ for every $1 \leqslant \alpha < \beta$ and the derivatives of $u \in U_\alpha$ $(u \in U_\alpha^b)$ satisfy the inequalities $(-1)^{k+1}u^{(k)} \geqslant 0$, where $k = 1, \ldots, n-1$ for integer n such that $n - 1 \leqslant \alpha < n$. The main advantage of using continua orders is given by their definitions in terms of moments. It is well known that portfolio returns exhibit heavy tails that do not always guarantee finiteness of the first moments. We apply α stochastic dominance orders to portfolios:

(1) with $\alpha \neq 1$ only if all portfolios X belong to the $L^{\alpha-1}$ space (i.e., $L^{\alpha-1} = \{X \mid E(|X|^{\alpha-1}) < +\infty\}$;

(2) when $\alpha = 1$ (first-order stochastic dominance), no regularity conditions on moments are needed.

Thus, one can rank the investor's choices by using orderings $\underset{\alpha}{>}$ with $\alpha \in (1, 2)$, even when the finite first moments cannot be guaranteed. The following definition considers orders that generalize the classic Rothschild–Stiglitz (R-S) order.

DEFINITION 2.2. We state that X *dominates* Y *in the sense of* α-(*bounded*) *R-S* (*strict*) *order* (α-(*bounded*) *R-S* (*strict*)) when

$$X \underset{\alpha}{\geqslant} Y \ \ (X \overset{b}{\underset{\alpha}{\geqslant}} Y, X \underset{\alpha}{>} Y, X \overset{b}{\underset{\alpha}{>}} Y)$$

and

$$-X \underset{\alpha}{\geqslant} -Y \ \ (-X \overset{-a}{\underset{\alpha}{\geqslant}} -Y, -X \underset{\alpha}{>} -Y, -X \overset{-a}{\underset{\alpha}{>}} -Y).$$

We remark that in the literature, the α-R-S order is also known as an α-*concave order* when α is an integer that is greater than or equal to 2. In particular, when $\alpha = 2$, we obtain the classic R-S order. Furthermore, a (bounded) R-S order is strictly linked to the moment order. The following corollary summarizes some of the main implications regarding R-S type orderings.

COROLLARY 2.1. *The following implications hold*:

(a) X α-(*bounded*) *R-S* (*strict*) Y *implies* X β-(*bounded*) *R-S* (*strict*) Y *for all* $\beta \geqslant \alpha$.

(b) *Suppose that* X *strictly* α-*R-S* Y *and the moments of* X *and* Y *through integer* n *are finite for* $n-1 < \alpha \leqslant n$. *Then*

$$\big(E(X), \dots, E(X^n)\big) \neq \big(E(Y), \dots, E(Y^n)\big) \quad \text{and} \quad E(X^k) < E(Y^k)$$

for the minimum even k *such that* $E(X^k) \neq E(Y^k)$. *In particular*, *if* X *and* Y *are random variables with finite first moments*, *then* X α-*R-S* Y *implies* $E(X) = E(Y)$.

(c) X α-(*bounded*) *R-S* (*strict*) Y *if and only if* $dX + c$ α-(*bounded*) *R-S* (*strict*) $dY + c$ *for every* $c \in R$ $d > 0$ *if and only if for every real* t *for all* $t \in \operatorname{supp}(X, Y)$

$$\Gamma(\alpha) F_X^{(\alpha)}(t) = E\big((t - X)_+^{\alpha-1}\big) \leqslant E\big((t - Y)_+^{\alpha-1}\big) = \Gamma(\alpha) F_Y^{(\alpha)}(t),$$

$$\Gamma(\alpha) \bar{F}_X^{(\alpha)}(t) = E\big((X - t)_+^{\alpha-1}\big) \leqslant E\big((Y - t)_+^{\alpha-1}\big) = \Gamma(\alpha) \bar{F}_Y^{(\alpha)}(t)$$

(*and at least one inequality is strict for some* t *when the respective orders are strict*).

(d) X α *R-S* Y *implies that*

$$E(|X_1 - X_2|^{\alpha-1}) \leqslant E(|X_1 - Y_1|^{\alpha-1}) \leqslant E(|Y_1 - Y_2|^{\alpha-1}),$$

and $E(|X - t|^{\alpha-1}) \leqslant E(|Y - t|^{\alpha-1})$ *for every real* t (*that is strict for some* t *when the* α-*R-S order is strict*), *where* Y_1, Y_2 *are independent copies of* Y, *and* X_1, X_2 *are independent copies of* X, *and even* X_1, Y_1 *are independent.*

P r o o f. Points (a) and (c) are a consequence of the α-R-S order (α bounded R-S order) definition. Point (b) is a consequence of the Fishburn [8] and O'Brien [22] necessary condition of moments expressed in the previous proposition; that is generally true only if we consider unbounded dominance orders. Point (d) is a consequence of point (b) of Proposition 2.2. ■

Clearly, α must be strictly greater than 1 in the definition, because $X \underset{1}{>} Y$ implies $-Y \underset{1}{>} -X$ and we cannot have $E(X) > E(Y)$ and $-E(X) > -E(Y)$. In addition, we can compare bounded random variables in the sense of α-R-S order only when $\alpha \geqslant 2$, as it follows from the next proposition that summarizes some of the most important implications relative to R-S type orders.

PROPOSITION 2.3. *The following implications hold*:

(a) *Assume Y belongs to L^p with $p > \alpha$. If X α-R-S Y and $E\left(|X|^r\right) = E\left(|Y|^r\right)$ for a given $r \in (\alpha - 1, p]$, then $F_X = F_Y$; otherwise X α-R-S strictly Y implies $E\left(|X|^r\right) < E\left(|Y|^r\right)$ for every $r \in (\alpha - 1, p]$. In particular, a random variable $X \notin L^p$ can never α-R-S dominate a random variable $Y \in L^p$.*

(b) *If X and Y are (below or above) bounded random variables with first moment finite, then there exists no $\alpha \in (1, 2)$ such that X α-(bounded) R-S strictly Y.*

(c) *If X and Y are symmetric with null mean, X α-(bounded) R-S Y if and only if $X \underset{\alpha}{\geqslant} Y$ ($X \overset{b}{\underset{\alpha}{\geqslant}} Y$).*

P r o o f. Point (a) generalizes Theorem 2.6 of Li and Zhu [15]. By the previous Lemma 2.1, we know that

$$|x|^r = \frac{1}{B(\alpha, r - \alpha + 1)} \int_0^{|x|} (|x| - y)^{\alpha - 1} y^{r - \alpha} dy$$

for every $r > \alpha - 1$. Then, as a consequence of the Fubini theorem, we get

$$\begin{aligned}
&B(\alpha, r - \alpha + 1) E\left(|X|^r\right) \\
&= \int_a^b \Big(\int_0^{|x|} (|x| - y)^{\alpha - 1} y^{r - \alpha} dy \Big) dF_X(x) \\
&= \int_0^b y^{r - \alpha} \Big(\int_y^b (x - y)^{\alpha - 1} dF_X(x) \Big) dy + \int_a^0 (-y)^{r - \alpha} \Big(\int_a^y (y - x)^{\alpha - 1} dF_X(x) \Big) dy \\
&= \Gamma(\alpha) \int_0^b y^{r - \alpha} \bar{F}_X^{(\alpha)}(y) dy + \Gamma(\alpha) \int_a^0 (-y)^{r - \alpha} F_X^{(\alpha)}(y) dy.
\end{aligned}$$

Thus,

$$B(\alpha, r-\alpha+1)E\left(|Y|^r-|X|^r\right)=\Gamma(\alpha)\int_0^b y^{r-\alpha}\big(\bar{F}_Y^{(\alpha)}(y)-\bar{F}_X^{(\alpha)}(y)\big)dy$$
$$+\Gamma(\alpha)\int_a^0(-y)^{r-\alpha}\big(F_Y^{(\alpha)}(y)-F_X^{(\alpha)}(y)\big)dy.$$

If X α-R-S Y and $E\left(|X|^r\right)=E\left(|Y|^r\right)$ with $r>\alpha-1$, we have $F_Y^{(\alpha)}=F_X^{(\alpha)}$ (i.e., $F_X=F_Y$); otherwise $E\left(|X|^r\right)<E\left(|Y|^r\right)$ for every $r>\alpha-1$ for which the finite r-th moment exists.

In order to prove (b), suppose X and Y are random variables bounded from below; then $X, Y>a>-\infty$. Since X and Y admit the finite first moment, we can suppose X dominates strictly Y in the sense of α-R-S order. However, under this assumption we have $E\left(X-a\right)=E\left(Y-a\right)$. By the previous point (a), if $\alpha\in[1,2)$ and X α-R-S Y (X α bounded R-S Y), then $E\big((X-a)^r\big)<E\big((Y-a)^r\big)$ for every $r>\alpha-1$, against $E\left(X-a\right)=E\left(Y-a\right)$. Similar considerations can be done when $X, Y<M$ because $-X,-Y>-M$. Thus point (b) follows, and so $\alpha\in[1,2)$ such that X α-R-S Y (X α bounded R-S Y) cannot exist when X and Y are bounded and they admit the finite first moment. If X and Y are symmetric with null mean $X=-X$ and $Y=-Y$, the point (c) holds true. ■

From the previous analysis, we deduce that the inequalities between absolute moments allow us to order portfolio uncertainty coherently to different types of investors. Another immediate consequence is the next corollary.

COROLLARY 2.2. *If in the market there exist two portfolios X and Y with the same mean and dispersion $E\big(\,|X-E(X)|^r\,\big)=E\big(\,|Y-E(Y)|^r\,\big)$, then either one portfolio is redundant* (*because it has the same distribution as the other*) *or the two portfolios are not comparable in the sense of* $(p+1)$*-R-S order for any* $p<r$.

According to an operational definition of the risk and uncertainty that is perceived by investors (see, e.g., Rachev et al. [28] and Holton [12]), the previous discussion suggests distinguishing the orderings with respect to (a) the uncertainty of different positions and (b) the investor's exposure to risk. Generally, R-S type orders serve to characterize the different degrees of portfolio uncertainty and for this reason are called *uncertainty orders*, while the orders (such as $\underset{\pm\alpha}{\geqslant}, \overset{b}{\underset{\alpha}{\geqslant}}$) derived by the monotonicity order (i.e., the order $X>Y$ implies that X dominates ($\underset{\pm\alpha}{\geqslant}, \overset{b}{\underset{\alpha}{\geqslant}}$) Y) also take into account the downside risk of portfolios and are called *risk orders*. Clearly, this first distinction could have an important impact for investors.

Specifically, to select the set of admissible choices which are coherent to a given category of investors, we can consider the direct risk measures $\rho(X)$ (associated with random wealth X) that are consistent with the order relation ($\underset{\pm\alpha}{\geqslant}$, $\overset{b}{\underset{\alpha}{\geqslant}}$, α-(bounded) R-S); that is, $\rho(X) \leqslant \rho(Y)$ if X dominates ($\underset{\pm\alpha}{\geqslant}$, $\overset{b}{\underset{\alpha}{\geqslant}}$, α-(bounded) R-S) Y. Typically, it follows that $\rho_{t,\alpha}(X) = E\big((t - X)_+^{\alpha-1}\big)$ is a measure consistent with $\underset{\alpha}{\geqslant}$ ($\overset{b}{\underset{\alpha}{\geqslant}}$) order for any fixed t (belonging to the support of all optimal portfolios). Similarly, the measures $\tilde{\rho}_\alpha(X) = E(|X_1 - X_2|^{\alpha-1})$ and $\tilde{\rho}_{t,\alpha}(X) = E(|t - X|^{\alpha-1})$ are consistent with α-R-S (α-bounded R-S) order for any fixed t (belonging to the support of all optimal portfolios) under the assumption that X_1, X_2 are independent copies of X. The measures consistent with risk orders are called *risk measures*, while the measures consistent with uncertainty orders are called *uncertainty measures*. Thus, as discussed by Ortobelli et al. [26], their use is different in portfolio choice problems.

Furthermore, we can order the choices considering reward instead of risk. According to the definition given by Rachev et al. [28] and De Giorgi [4], we assume a reward measure to be any functional v defined on portfolio returns that is isotonic with respect to a given stochastic risk order (for example: $\underset{\alpha}{\geqslant}$, $\overset{b}{\underset{\alpha}{\geqslant}}$). Thus, when a given category of investors (e.g., non-satiable, non-satiable risk averse) prefers X to Y, then $v(X) \geqslant v(Y)$. On the other hand, Rachev et al. [28] and Biglova et al. [3] have shown that the use of a reward-risk ratio could be important not only from a computational point of view, but also because it takes into account portfolio diversification. Any consideration that we do for measures consistent with some risk orderings can be extended to reward measures considering a maximization problem instead of a minimization problem. That is, if $\rho(X)$ is a risk measure consistent with a risk ordering, then $-\rho(X)$ is a reward measure isotonic with the same order. Thus, if we characterize the consistency with respect to risk orderings (say $\underset{\alpha}{\geqslant}$, $\overset{b}{\underset{\alpha}{\geqslant}}$), we also implicitly characterize isotonicity. For this reason, in the following we place much more emphasis on the consistency with a given order.

2.2. Inverse stochastic dominance. Similarly to classic stochastic dominance rules, we can describe stochastic dominance rules based on the left inverse of F_X (namely, F_X^{-1}) given by

$$F_X^{-1}(p) = \inf\{x : \Pr(X \leqslant x) = F_X(x) \geqslant p\} \qquad \text{for all } p \in (0, 1]$$

and $F_X^{-1}(0) = \lim_{p \searrow 0} F_X^{-1}(p)$. In particular, Muliere and Scarsini [19] have defined inverse stochastic dominance order as follows: we say that X *strictly dominates Y with respect to n (integer) inverse order stochastic dominance* ($X \underset{-n}{>} Y$) if and only if

$$F_X^{(-n)}(t) = \int_0^t F_X^{(-n+1)}(u)du \geqslant F_Y^{(-n)}(t) = \int_0^t F_Y^{(-n+1)}(u)du \quad \text{for all } t \in [0,1],$$

where we assume $F_X^{(-1)} = F_X^{-1}$. As for integer stochastic orders, even the above dual stochastic orders can be easily extended in continuous terms. Let us consider the unique completion of the σ-finite positive measure associated with F_X^{-1}, which on the half open intervals of the forms $[a, b) \subseteq [0, 1]$ is given by

$$\mu_X\big([a,b)\big) = F_X^{-1}(b) - F_X^{-1}(a) = \int_a^b dF_X^{-1}(p).$$

Then we can define the α dual functions:

$$F_X^{(-1)}(p) = F_X^{-1}(p) \quad \text{for all } p \in [0,1],$$

$$F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)} \int_0^p (p-u)^{\alpha-1} dF_X^{-1}(u) \quad \text{for all } p \in [0,1],\ \alpha \neq 1, \tag{2.6}$$

which are continuous for every $\alpha > 1$ and left continuous for $\alpha \leqslant 1$. Moreover, the functions $\bar{F}_X^{(-1)}(p) = -F_X^{(-1)}(p)$ for all $p \in [0, 1]$ and

$$\bar{F}_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)} \int_{p^+}^1 (u-p)^{\alpha-1} dF_X^{-1}(u) \quad \text{for all } p \in [0,1],\ \alpha \neq 1,$$

are continuous for every $\alpha > 1$, left continuous for $\alpha = 1$ and right continuous for $\alpha < 1$. In particular, when X is a continuous random variable, it follows that $\bar{F}_X^{(-\alpha)}(p) = F_{-X}^{(-\alpha)}(1-p)$ for all $\alpha > 0$. Moreover, as a consequence of Lemma 2.1 we obtain

$$F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha - v)} \int_0^p (p-u)^{\alpha-v-1} F_X^{(-v)}(u)du \quad \text{for all } \alpha > v > 0,$$

$$\bar{F}_X^{(-\alpha)}(p) = \begin{cases} \dfrac{1}{\Gamma(\alpha - v)} \displaystyle\int_{p^+}^1 (u-p)^{\alpha-v-1} \bar{F}_X^{(-v)}(u)du \\ \qquad\qquad \text{for all } \alpha > v \geqslant 1 \text{ or } 1 > \alpha > v > 0, \\ \displaystyle\lim_{p_n \nearrow p} \frac{1}{\Gamma(\alpha - v)} \int_{p_n}^1 (u-p_n)^{\alpha-v-1} \bar{F}_X^{(-v)}(u)du \\ \qquad\qquad \text{for all } \alpha > 1 > v > 0. \end{cases}$$

In particular, when $\alpha = 2$, we obtain the absolute generalized Lorenz curve

$$F_X^{(-2)}(p) = L_X(p) = \int_0^p F_X^{-1}(t)dt$$

(that is formally different from the relative Lorenz curve often used in income inequality[1]). Thus the following definition extends the previous dual orders to continua orders.

DEFINITION 2.3. For every $\alpha > 0$, we say that X *dominates* Y *with respect to the* α *dual* (also called *inverse*) *stochastic order* ($X \underset{-\alpha}{\geqslant} Y$) iff

$$F_X^{(-\alpha)}(t) \geqslant F_Y^{(-\alpha)}(t) \quad \text{for all } t \in [0,1],$$

and we say that X *strictly dominates* Y *with respect to the* α *dual order* ($X \underset{-\alpha}{>} Y$) iff

$$X \underset{-\alpha}{\geqslant} Y \quad \text{and} \quad F_X \neq F_Y.$$

We say that X *dominates* Y *in the sense of the dual* α*-R-S order* (*strict*) (*dual* α*-R-S* (*strict*)) if

$$X \underset{-\alpha}{\geqslant} Y (X \underset{-\alpha}{>} Y) \quad \text{and} \quad -X \underset{-\alpha}{\geqslant} -Y (-X \underset{-\alpha}{>} -Y).$$

Similarly, we can define the *survival order*, that is,

$$X \underset{\text{sur}-\alpha}{\geqslant} Y \quad \text{iff} \quad \bar{F}_X^{(-\alpha)}(t) \leqslant \bar{F}_Y^{(-\alpha)}(t)$$

for every t belonging to $[0,1]$. Since for $\alpha > 1$ we get $\bar{F}_X^{(-\alpha)}(p) = F_{-X}^{(-\alpha)}(1-p)$, the results obtained for survival dual orders (with $\alpha > 1$) are equivalent to those obtained for orders applied to the opposite of the random variables. From this definition we infer that $F_X^{(-v)}(p)$ is a reward measure for any p belonging to $(0,1)$. As for the α stochastic orders, we can prove similar properties for the dual stochastic orders. In particular, it is well known that $\underset{1}{\geqslant}$ and $\underset{2}{\geqslant}$ orders are equivalent to the respective $\underset{-1}{\geqslant}$ and $\underset{-2}{\geqslant}$ orders. Therefore, all the implications which are valid for $\underset{1}{\geqslant}$, $\underset{2}{\geqslant}$ ($\overset{b}{\underset{1}{\geqslant}}$, $\overset{b}{\underset{2}{\geqslant}}$) and 2-(bounded) R-S orders are still valid for the equivalent orders $\underset{-1}{\geqslant}$, $\underset{-2}{\geqslant}$ and dual 2-R-S orders. However, integer stochastic dominance orders greater than two are different by the respective dual orders (see, among others, Muliere and Scarsini [19]). This is logical because the inverse stochastic order is defined only

[1] The relative Lorenz curve is given by $L_X(p)/E(X)$; see Arnold [1], Ogryczak and Ruszczynski [23].

on the support of the random variables (as $\overset{b}{\underset{\alpha}{\geqslant}}$ order but differently by $\underset{\alpha}{\geqslant}$ order). Thus there probably exists a correspondence between $\overset{b}{\underset{\alpha}{\geqslant}}$ and dual orders, which will be the subject of future research.

On the other hand, we observe that inverse stochastic orders previously defined can be extended to unbounded inverse stochastic orders as follows. Suppose that either $|F_X^{(-1)}(0)| < \infty$ or $|F_X^{(-1)}(1)| < \infty$ for X belong to a given class of random variables Λ. Then we extend $F_X^{(-1)}$ on the whole real line R assuming $F_X^{(-1)}(u) = F_X^{(-1)}(0)$ for all $u \leqslant 0$ and $F_X^{(-1)}(t) = F_X^{(-1)}(1)$ for all $t \geqslant 1$. Moreover, we say that X *dominates* Y *with respect to the unbounded* α *inverse stochastic order* (unbounded $X \underset{-\alpha}{\geqslant} Y$) iff

$$F_X^{(-\alpha)}(u) \geqslant F_Y^{(-\alpha)}(u) \quad \text{for every } u \in R,$$

where

$$F_X^{(-\alpha)}(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u^-} (u-t)^{\alpha-1} dF_X^{(-1)}(t).$$

Many of the considerations done for stochastic dominance orders can be repeated for dual orders, and in the next remark we summarize the main properties of these orders.

REMARK 2.2. *The following implications hold*:

(1) *Unbounded* $X \underset{-\alpha}{\geqslant} Y$ *implies* $X \underset{-\alpha}{\geqslant} Y$. *In addition, for every* $\beta \geqslant \alpha$, (*unbounded*) $X \underset{-\alpha}{\geqslant} Y$ *implies* (*unbounded*) $X \underset{-\beta}{\geqslant} Y$, *and* X *dual* (*unbounded*) α-*R-S* Y *implies* X *dual* (*unbounded*) β-*R-S* Y.

(2) $X \underset{-\alpha}{\geqslant} Y$ *if and only if* $cX + t \underset{-\alpha}{\geqslant} cY + t$ *for every* $t \in R$, $c > 0$, $\alpha > 0$. $X \underset{-\alpha}{\geqslant} Y$ *with* $\alpha > 1$ *implies* $X_-^{(M)} \underset{-\alpha}{\geqslant} Y_-^{(M)}$ *for any given* $M \in R$.

(3) *For every* $\alpha > 1$ *and for every* X, Y *belonging to the set*

$$\Lambda_{(\alpha)} := \{X \mid |F_X^{(-\alpha)}(x)| < \infty \text{ for all } x \in (0,1)\},$$

it follows that $X \underset{-\alpha}{\geqslant} Y$ *if and only if*

$$\int_0^1 \phi(x) dF_X^{-1}(x) \leqslant \int_0^1 \phi(x) dF_Y^{-1}(x) \quad \text{for every } \phi \in V^\alpha,$$

where

$$V^{\alpha} = \big\{\phi(x) = -\int\limits_{x^+}^{1} (s-x)^{\alpha-1} d\tau(s) - k(1-x)^{\alpha-1} \mid k \geqslant 0;$$

τ is a σ-finite positive measure such that for all $X \in \Lambda_{(\alpha)}$
the function $|s-x|^{\alpha-1}$ is $d\tau(s) \times dF_X^{-1}(x)$ integrable in $[0,1] \times [0,1]\big\}$.

(4) *For every $\alpha > 1$ and for every $X, Y \in \Lambda_{(\alpha)} = \{X \mid |F_X^{(-\alpha)}(x)| < \infty$ for all $x \in R\}$, it follows that unbounded $X \underset{-\alpha}{\geqslant} Y$ if and only if*

$$\int\limits_{-\infty}^{+\infty} \phi(x) dF_X^{-1}(x) \leqslant \int\limits_{-\infty}^{+\infty} \phi(x) dF_Y^{-1}(x) \quad \textit{for every } \phi \in UV^{\alpha},$$

where

$$UV^{\alpha} = \big\{\phi(x) = -\int\limits_{x^+}^{+\infty} (s-x)^{\alpha-1} d\tau(s) \mid \tau \textit{ is a } \sigma\textit{-finite positive measure such that}$$

for all $X \in \Lambda_{(\alpha)}$ the function $|s-x|^{\alpha-1}$ is $d\tau(s) \times dF_X^{-1}(x)$ integrable in $R^2\big\}$.

(5) *If $X \underset{-\alpha}{\geqslant} Y$, then for any integer $k \geqslant \alpha - 1$ the inequality*

$$E(\min_{1 \leqslant i \leqslant k} X_i) \geqslant E(\min_{1 \leqslant i \leqslant k} Y_i)$$

holds, where X_i, Y_i $(i = 1, \ldots, k)$ are i.i.d. copies of X and Y, respectively.

P r o o f. While the first three points follow by the previous discussion, implications (4) and (5) are a logical consequence of the analysis proposed by Muliere and Scarsini [19]. ■

From the above discussion it follows that there exist many different ways to discriminate between the choices available to investors. We distinguish between orders and their dual/survival orders, bounded and unbounded orders, and risk and uncertainty orders. Moreover, there exists a strong connection among orderings and risk/uncertainty measures that will be more thoroughly treated in the next section.

3. NEW MEASURES FOR ORDERINGS AND PROBABILITY FUNCTIONALS

Most of portfolio theory is based on minimizing a distance from a benchmark or minimizing potential possible losses while maintaining constant some portfolio characteristics. As observed by Rachev et al. [28], these problems can be reformulated from the point of view of the theory of probability metrics. In particular, we are generally interested in probability functionals $\mu : \Lambda \times \Lambda \to R$ (where Λ is a non-empty space of real-valued random variables defined on $(\Omega, \Im, P)$) satisfying the following property for any pair of random variables X, Y:

IDENTITY PROPERTY. $f(X) = f(Y) \Leftrightarrow \mu(X,Y) = 0$, where $f(X)$ identifies some characteristics of the random variable X.

From this property we can distinguish among three main groups of probability functionals (namely, primary, simple, and compound) depending on certain modifications of the identity property (see Rachev [27]). *Compound probability functionals* identify the random variable almost surely (i.e., for any pair of random variables X, Y: $\mu(X,Y) = 0 \Leftrightarrow P(X = Y) = 1$). *Simple probability functionals* identify the distribution (i.e., for any pair of random variables X, Y: $\mu(X,Y) = 0 \Leftrightarrow F_X = F_Y$). *Primary probability functionals* determine only some random variable characteristics. Typically, with respect to the portfolio selection problem, the two probability functionals μ studied are those that identify:

(1) *The uncertainty of the random variable in a given absolute moment*.

Thus, we can say that some portfolios are *equivalent in uncertainty* if they present the same dispersion that can be measured in different ways, see Ortobelli et al. [26]. For example, we can consider equivalent in uncertainty portfolios with:

- the same distance by a given benchmark Z,

$$\mu(X,Y) = 0 \Leftrightarrow d(X,Z) = d(Y,Z),$$

where *d* measures a distance between the random variable and the benchmark Z;

- the same level of concentration valued with an opportune moment *p*, i.e.

$$\mu(X,Y) = 0 \Leftrightarrow E(|X_1 - X|^p) = E(|Y_1 - Y|^p),$$

and where X_1 is an independent copy of X and Y_1 is an independent copy of Y.

(2) *The losses in distributional tail behavior*.

Thus, for example we can assume *equivalent in losses* (*risk*) two investments that present

- the same distributional tail

$$\mu(X,Y) = 0 \Leftrightarrow F_X(x) = F_Y(x) \quad \text{for all } x \in (-\infty, t]$$

for a given t;

- the same power of the tail valued on the left tails with an opportune moment,

$$\mu(X,Y) = 0 \Leftrightarrow E\left((t - X)_+^p\right) = E\left((t - Y)_+^p\right)$$

for a given threshold $t \in R$.

Further extensions that describe primary, simple, and compound probability metrics as tracking error measures can be found in Stoyanov et al. [33] and Ortobelli et al. [25].

3.1. FORS orderings. One of the principal problems in economics is the ordering of choices in the face of uncertainty. Basically, any observer can deduce the decision makers' preferences from their behavior in the market. Starting from this logical deduction, utility theory classifies the optimal choices of different categories of market agents (for example, risk-averse, non-satiable, non-satiable risk averse) under ideal market conditions. In particular, the fundamentals of utility theory under uncertainty conditions have been developed by von Neumann and Morgenstern [37]. Several improvements and further advancements of the theory have been proposed, even in recent years; see, among others, Machina [17], Yaari [38], Gilboa and Schmeidler [11], and Maccheroni et al. [16]. Roughly speaking, in utility theory the ordering of uncertain choices begins with the selection of a finite number of axioms characterizing the preferences of a given class of market agents.

The second step of the theory involves representing the preferences of market agents using "utility functionals" that summarize the decision makers' behavior. Clearly, there exists a correspondence among the orderings of utility functionals, the orderings of preferences, and the orderings of random variables. Thus, when utility functionals are characterized, it is possible to identify the different categories of market agents. Consequently, we can also identify the optimal choices for a given class of market agents when we order some utility functionals. In particular, we define as efficient, for a given category of market operators, all the admissible choices that cannot be preferred (dominated) by all the agents in the same category. Moreover, there exists a correspondence between utility functionals and probability functionals. Therefore, in order to capture the agents' behavior, we propose to study orderings among probability functionals which are induced by orderings among preferences.

According to the definition of probability functionals (see Rachev [27]), we want to discuss the main relevant properties of a probability functional with respect to the portfolio selection problem. It is well known that the most important property that characterizes any probability functional μ associated with a portfolio choice problem is the consistency with a stochastic order, see Ortobelli et al. [26]. In terms of probability functionals, the *consistency* is defined as: X dominates Y with respect to a given order of preferences $\succ$ implies $\mu(X, Z) \leqslant \mu(Y, Z)$ for a fixed arbitrary benchmark Z.

We define a FORS measure induced by order $\succ$ as any probability functional $\mu : \Lambda \times \Lambda \to R$ that is consistent with a given order of preferences $\succ$. The order of preferences $\succ$ could be characterized either with (a) some axioms that identify the decision makers' preferences (as in utility theory); or with (b) an order that identifies the preferences of a particular category of investors characterized by the parameter α, such as orders $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$, $\underset{-\alpha}{>}$, unbounded $\underset{-\alpha}{>}$ and (dual) α-(bounded) R-S order. In case (b), we simply call α-FORS order, the order of preference $\succ$

and α-FORS measure induced by the α-FORS order any probability functional $\mu : \Lambda \times \Lambda \to R$ consistent with the given order of preferences.

Observe that in the definition of consistency, no rule relative to the benchmark Z is described. As a matter of fact, the benchmark Z is a fixed random variable that depends on the order of preferences $\succ$. Therefore, as a subclass of probability functionals consistent with an order of preference we can consider all the risk measures $\mu : \Lambda \to R$. In particular, the recent literature in financial economics has highlighted the importance of some particular properties of risk measures; see, among others, Artzner et al. [2], Frittelli and Rosazza Gianin [10], Föllmer and Sheid [9], and Ortobelli et al. [26]. We recall that a convex measure $\mu(X)$ valued on a family of random variables $X \in \Lambda$ is:

1. *monotone*: for every $X, Y \in \Lambda$, $X \geqslant Y \Rightarrow \mu(X) \leqslant \mu(Y)$;
2. *translation invariant*[2]: for all $X \in \Lambda$ and $m \in R$, $\mu(X + m) = \mu(X) - m$;
3. *convex*: for all $X, Y \in \Lambda$ and for all $a \in [0, 1]$,

$$\mu\big(aX + (1 - a)Y\big) \leqslant a\mu(X) + (1 - a)\mu(Y).$$

If additionally we even consider positive homogeneity,

4. *positive homogeneous*: for all $\alpha \geqslant 0$ and for all $X \in \Lambda$, $\mu(\alpha X) = \alpha\mu(X)$,

then, we have a coherent static risk measure.

Thus any coherent risk measure is a FORS measure $\mu : \Lambda \to R$ induced by the monotonic order, i.e., for all $X, Y \in \Lambda$, $X > Y$ P-almost surely implies $\mu(X) \leqslant \mu(Y)$.

DEFINITION 3.1. We call a *convex α-FORS measure* any translation invariant, convex probability functional $\mu : \Lambda \to R$ that is consistent with an α-FORS order. We call a *coherent α-FORS measure* any translation invariant, convex, and positive homogeneous probability functional $\mu : \Lambda \to R$ that is consistent with an α-FORS order.

Although in many cases convex/coherent risk measures are convex/coherent FORS measures, this definition better specifies the consistency. For example, for every $\alpha \geqslant 1$ and for every $\beta \in (0, 1)$ the measure

$$\frac{-\Gamma(\alpha + 1)}{\beta^{\alpha}} F_X^{(-(\alpha+1))}(\beta)$$

is a coherent $(\alpha + 1)$-FORS measure consistent with $\underset{-(\alpha+1)}{\geqslant}$ order (see Ortobelli et al. [26]). However, this measure is not necessarily consistent with $\underset{-(\gamma+1)}{\geqslant}$ when

[2] Observe that there exist several alternative definitions of translation invariance property associated with financial random variables (see Ortobelli et al. [26]). Since the translation invariance is often used to value the risk of a random variable, some authors apply the property to the opposite of the underlying random variable. In this case, the above property becomes: $\mu(X + m) = \mu(X) + m$.

$\gamma > \alpha$ (i.e., it is not a coherent $(\gamma+1)$-FORS measure). Among the typical FORS functionals we can consider the following ones:

1. $-F_X^{(-\alpha)}(p)$ for a fixed benchmark $p \in (0,1)$ (is induced by $\underset{-\alpha}{>}$ order);

2. $F_X^{(\alpha)}(t)$ for a fixed benchmark $t \in R$ (is induced by $\underset{\alpha}{>}$ order);

3. $\tilde{\rho}_{t,\alpha}(X) = E(|t-X|^{\alpha-1})$ for a fixed benchmark $t \in R$ (is induced by α-R-S order);

4. $\tilde{\rho}_{\alpha}(X) = E(|X-X_1|^{\alpha-1})$ for the benchmark X_1 that is an independent copy of X (is induced by α-(bounded) R-S order).

As for the previous ordering analysis, we deduce that there exist two types of FORS measures:

• *measures of risk* (tails, losses) which are induced by the monotonicity order or by all the orderings of tails such as $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$, $\underset{-\alpha}{>}$, that we call *FORS risk measures* (that are measures of reward if we multiply the functions by -1);

• *measures of uncertainty* (concentration, dispersion) which are induced by orderings of uncertainty such as (dual) α-(bounded) R-S orders, that we call *FORS uncertainty measures*.

Similarly we can extend the previous definition to reward measures isotonic to orderings. To do so we assume that any probability functional μ associated with a portfolio choice problem satisfies the following property:

(b-bis) (isotonicity) X dominates Y with respect to a risk ordering $\succ$ implies $\mu(X,Z) \geqslant \mu(Y,Z)$ for a fixed arbitrary benchmark Z.

Then the probability functional μ is called a *FORS reward measure* induced by the risk order $\succ$.

Clearly, any consideration done for FORS risk measures can be easily extended to FORS reward measures. Moreover, all the above examples of FORS functionals induced from a given ordering of preference $\succ$ are parametric. However, under the opportune hypotheses, we can also say the converse. As a matter of fact, one can develop many other kinds of orderings using the fractional integral in the following way.

DEFINITION 3.2. Assume $\rho_X : [a,b] \to \bar{R}$ (with $-\infty \leqslant a < b \leqslant +\infty$) is a bounded variation function for every random variable X belonging to a given class Λ. Furthermore, assume that ρ_X is a simple probability functional over the class Λ (i.e., for all $X, Y \in \Lambda$, $\rho_X = \rho_Y \Leftrightarrow F_X = F_Y$) and suppose that, for any fixed $\lambda \in [a,b]$, $\rho_X(\lambda)$ is a FORS risk measure induced by a risk ordering $\succ$. Then, for every $\alpha > 0$ and for all $X, Y \in \Lambda_{(\alpha)}$, where

$$\Lambda_{(\alpha)} = \left\{ X \in \Lambda \,\Big|\, \Big| \int_a^b |t|^{\alpha-1}\, d\rho_X(t) \Big| < \infty \right\},$$

we say that X *dominates* Y *in the sense of* α*-FORS risk ordering induced by* $\succ$

(in symbols, $X \underset{\succ,\alpha}{\mathrm{FORS}} Y$) if and only if

$$\rho_{X,\alpha}(u) \leqslant \rho_{Y,\alpha}(u) \quad \text{for all } u \in [a,b],$$

where

$$\rho_{X,\alpha}(u) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_a^{u^-} (u-t)^{\alpha-1} d\rho_X(t) & \text{if } \alpha > 0, \alpha \neq 1, \\ \rho_X(u) & \text{if } \alpha = 1. \end{cases}$$

We call the new class of orderings *FORS risk orderings induced by* $\succ$, and we call ρ_X a *FORS measure associated with the FORS ordering* of random variables belonging to Λ.

In contrast to classic stochastic dominance orders, it could happen that

$$X \underset{\succ,\alpha}{\mathrm{FORS}} Y \quad \text{with } \alpha \in (0,1)$$

even when the random variables X and Y are bounded and continuous on the extremes of their support. Similarly, we can define FORS uncertainty orderings.

DEFINITION 3.3. We say that *X dominates Y in the sense of α-FORS uncertainty ordering induced by* $\succ$ (we simply write $X \underset{\succ,\mathrm{unc}\ \alpha}{\mathrm{FORS}} Y$) if and only if

$$\int\limits_a^x (x-s)_+^{\alpha-1}\, d\rho_{\pm X}(s) \leqslant \int\limits_a^x (x-s)_+^{\alpha-1}\, d\rho_{\pm Y}(s) \quad \text{for all } x \in [a,b],$$

i.e. when

$$X \underset{\succ,\alpha}{\mathrm{FORS}} Y \quad \text{and} \quad -X \underset{\succ,\alpha}{\mathrm{FORS}} -Y.$$

Given a FORS ordering, then it is possible to define a survival ordering as follows:

$$\bar{\rho}_{X,\alpha}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_{t^+}^{b} (u-t)^{\alpha-1} d\rho_X(u) & \text{if } \alpha > 0,\ \alpha \neq 1, \\ -\rho_X(t) & \text{if } \alpha = 1, \end{cases}$$

$$\bar{\rho}_{X,\alpha}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha - v)} \int\limits_{t^+}^{b} (u-t)^{\alpha-v-1} \bar{\rho}_{X,v}(u) du & \\ & \text{for all } \alpha > v \geqslant 1 \text{ or } 1 > \alpha > v > 0, \\ \lim\limits_{t_n \nearrow t} \dfrac{1}{\Gamma(\alpha - v)} \int\limits_{t_n^+}^{b} (u-t_n)^{\alpha-v-1}\, \bar{\rho}_{X,v}(u) du & \text{for all } \alpha > 1 > v > 0, \end{cases}$$

and we say that

$$X \underset{\succ,\text{sur }\alpha}{\text{FORS}} Y \text{ iff } \bar{\rho}_{X,\alpha}(t) \leqslant \bar{\rho}_{Y,\alpha}(t) \text{ for every } t \in [a,b].$$

However, in this case, we cannot generally say that the results obtained for survival orders are equivalent to those obtained for orders applied to the opposite of the random variables. Thus the survival FORS ordering is an alternative to the original one. Note that if we assume in Definition 3.2 that ρ_X is a primary (instead of simple) probability functional induced by $\succ$, then the probability functionals

$$\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_a^{u^-} (u-t)^{\alpha-1} d\rho_X(t)$$

defined for $\alpha > 1$ are again FORS measures induced by $\succ$. In addition, if σ_X is a FORS probability functional induced by a given FORS ordering $\underset{\succ,v}{\text{FORS}}$, then σ_X is again a FORS measure induced by the order $\succ$.

For any FORS risk ordering induced by $\succ$, we can easily define an inverse (dual) ordering if the FORS measure ρ_X is monotone. In this case, we consider the left inverse of ρ_X (i.e., $\rho_X^{-1}(x) = \inf\{u \in [a,b] : \rho_X(u) \geqslant x\}$ for any real x belonging to the value domain of ρ_X). However, many of the extensions we have observed for stochastic dominance order and its dual are still valid for FORS orderings as described in the following remark.

REMARK 3.1. *The following implications hold for a FORS ordering of a random variables class* Λ:

(1) *For every* $\alpha > v > 0$, $X \underset{\succ,v}{\text{FORS}} Y$ *implies* $X \underset{\succ,\alpha}{\text{FORS}} Y$, *and we can write*

$$\rho_{X,\alpha}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha-v)} \displaystyle\int_a^{t^-} (t-u)^{\alpha-v-1} \rho_{X,v}(u) du & \\ \qquad\qquad \textit{for all } \alpha > v \geqslant 1 \textit{ or } 1 > \alpha > v > 0, \\ \displaystyle\lim_{t_n \searrow t} \frac{1}{\Gamma(\alpha-v)} \int_a^{t_n} (t_n-u)^{\alpha-v-1} \rho_{X,v}(u) du & \textit{for all } \alpha > 1 > v > 0. \end{cases}$$

(2) *For any monotone increasing FORS measure* ρ_X *associated with a FORS ordering, the left inverse* ρ_X^{-1} *is a FORS reward measure and* $-\rho_X^{-1}$ *is itself a FORS ordering induced by* $\succ$.

(3) *Suppose* $|\rho_X(b)| < \infty$, $|\rho_X(a)| < \infty$ *for every* X *belonging to* Λ. *Then we can extend* ρ_X *on all the real line* R *assuming* $\rho_X(u) = \rho_X(a)$ *for all* $u \leqslant a$ *and* $\rho_X(u) = \rho_X(b)$ *for all* $u \geqslant b$. *Moreover, we say that* X unbounded $\underset{\succ,\alpha}{\text{FORS}}$ dominates Y *iff* $\rho_{X,\alpha}(u) \leqslant \rho_{Y,\alpha}(u)$ *for every* $u \in R$, *where we define*

$$\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u^-} (u-t)^{\alpha-1} d\rho_X(t) \quad \textit{for all } u \in R.$$

If ρ_X *is monotone, then unbounded* $\underset{\succ,\alpha}{\text{FORS}}$ *order implies* $\underset{\succ,\alpha}{\text{FORS}}$ *order.*

P r o o f. It follows from the previous definitions and discussions. ■

In addition, an equivalent formulation of FORS orderings is given by the following corollary that generalizes the representation of orderings using utility functionals.

COROLLARY 3.1. *Suppose ρ_X is a FORS measure associated with a FORS ordering $\succ$ on a given class of random variables X belonging to Λ. Then, given $X, Y \in \Lambda_{(\alpha)}$, $X \underset{\succ,\alpha}{\text{FORS}} Y$ if and only if*

$$\int_a^b \phi(u) d\rho_{X,1}(u) \geqslant \int_a^b \phi(u) d\rho_{Y,1}(u)$$

for every ϕ belonging to

$$W^\alpha = \Big\{\phi(x) = -\int_{x^+}^b (s-x)^{\alpha-1} d\tau(s) - k(b-x)^{\alpha-1} \mid k \geqslant 0, k = 0 \text{ if } b = \infty;$$
$$\tau \text{ is a } \sigma\text{-finite positive measure such that for all } X \in \Lambda_{(\alpha)}$$
$$\text{the function } |s-x|^{\alpha-1} \text{ is } d\tau(s) \times d\rho_X(x) \text{ integrable in } [a,b] \times [a,b]\Big\}.$$

Moreover, for every $1 \leqslant \alpha < v$, $\phi_v \in W^v$ if and only if there exists a function $\phi_\alpha \in W^\alpha$ such that

$$\phi_v(x) = \int_{x^+}^b (s-x)^{v-\alpha-1} \phi_\alpha(s) ds.$$

P r o o f. The proof of this corollary is analogous to the proof given by Muliere and Scarsini [19], Fishburn [6], [7] and Müller [20] with some little differences. In particular, observe that if $X \underset{\succ,\alpha}{\text{FORS}} Y$, then $\rho_{X,\alpha}(u) \leqslant \rho_{Y,\alpha}(u)$ for every u belonging to $[a,b]$. Thus,

$$\int_a^b (b-s)^{\alpha-1} d\rho_X(s) \leqslant \int_a^b (b-s)^{\alpha-1} d\rho_Y(s)$$

and

$$\int_a^b \int_a^{u^-} (u-s)^{\alpha-1} d\rho_X(s) d\tau(u) \leqslant \int_a^b \int_a^{u^-} (u-s)^{\alpha-1} d\rho_Y(s) d\tau(u).$$

From the Fubini–Tonelli theorem, this is equivalent to the inequality

$$\int_a^b \phi(u) d\rho_X(u) \geqslant \int_a^b \phi(u) d\rho_Y(u),$$

where $\phi(x) = -\int_{x^+}^{b}(s-x)^{\alpha-1}d\tau(s) - k(b-x)^{\alpha-1}$. Conversely, let us consider

$$\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)}\int_{a}^{u^-}(u-s)^{\alpha-1}d\rho_X(s) = \int_{a}^{b}\phi_{(u)}(s)d\rho_X(s),$$

where

$$\phi_{(u)}(s) = \frac{(u-s)^{\alpha-1}I_{[a,u)}(s)}{\Gamma(\alpha)} = \int_{s^+}^{b}(z-s)^{\alpha-1}d\tau_{(u)}(z)$$

and

$$\tau_{(u)}(y) = \frac{I_{(u,b]}(y)}{\Gamma(\alpha)}.$$

Clearly, for every $u \in [a,b]$, $-\phi_{(u)}(s) \in W^\alpha$, and the inequality

$$\int_{a}^{b}\big(-\phi_{(u)}(s)\big)d\rho_X(s) \geqslant \int_{a}^{b}\big(-\phi_{(u)}(s)\big)d\rho_Y(s)$$

implies that for all $u \in [a,b]$, $\rho_{X,\alpha}(u) \leqslant \rho_{Y,\alpha}(u)$, i.e., $X \underset{\succ,\alpha}{\mathrm{FORS}} Y$. Moreover, as a consequence of Lemma 1, for every $\alpha < v$, we have $\phi_v \in W^v$ if and only if

$$\begin{aligned}
\phi_v(x) &= -\int_{x^+}^{b}(s-x)^{v-1}\,d\tau_v(s) - k\,(b-x)^{v-1}\\
&= -\int_{x^+}^{b}\frac{(s-x)^{v-\alpha-1}}{B\,(v-\alpha,\alpha)}\Big(\int_{s^+}^{b}(y-s)^{\alpha-1}\,d\tau_v(y) - k\,(b-s)^{\alpha-1}\Big)ds\\
&= \int_{x^+}^{b}(s-x)^{v-\alpha-1}\,\phi_\alpha(s)ds,
\end{aligned}$$

where

$$\phi_\alpha(s) = -\int_{s^+}^{b}(y-s)^{\alpha-1}\,d\tau_\alpha(y) - \frac{k\,(b-s)^{\alpha-1}}{B\,(v-\alpha,\alpha)}, \qquad \tau_\alpha(y) = \frac{\tau_v(y)}{B(\alpha,v-\alpha)},$$

and $\phi_\alpha \in W^\alpha$. ■

Moreover, as follows from the proposition below, even some of the moments properties we have verified for the stochastic dominance orders can be replaced for FORS orderings.

PROPOSITION 3.1. *Suppose $\rho_X : [a,b] \to \bar{R}$ is a FORS measure associated with a FORS ordering $\succ$ for a given class of random variables X belonging to Λ. Then the following implications hold for any opportune pair of random variables X and Y belonging to Λ:*

(a) $X \underset{\succ,\alpha}{\text{FORS}} Y$ $(\alpha > 1)$ *implies the following relations for any increasing and invertible function* $H\colon \operatorname{supp}(X,Y) \to [a,b]$ *such that* $|H(z)-x|^{\alpha-1}$ *is* $dF_Z(z) \times d\rho_X(x)$ *integrable for* Z *equal either to* X *or to* Y:

$$\begin{aligned}
E\Big(\rho_{X,\alpha}\big(H(X)\big)\Big) &= \int_a^b E\Big(\big(H(X)-s\big)_+^{\alpha-1}\Big)d\rho_X(s)\\
&\leqslant \int_a^b E\Big(\big(H(X)-s\big)_+^{\alpha-1}\Big)d\rho_Y(s) = E\Big(\rho_{Y,\alpha}\big(H(X)\big)\Big),\\
E\Big(\rho_{X,\alpha}\big(H(Y)\big)\Big) &= \int_a^b E\Big(\big(H(Y)-s\big)_+^{\alpha-1}\Big)d\rho_X(s)\\
&\leqslant \int_a^b E\Big(\big(H(Y)-s\big)_+^{\alpha-1}\Big)d\rho_Y(s) = E\Big(\rho_{Y,\alpha}\big(H(Y)\big)\Big).
\end{aligned}$$

In particular, when $\operatorname{supp}(X,Y) = [c,d]$, *we can take*

$$H(x) = \frac{x-c}{d-c}(b-a)+a.$$

(b) *If* $X \underset{\succ,sur\alpha}{\text{FORS}} Y$ *and* $X \underset{\succ,\alpha}{\text{FORS}} Y$ (*i.e.*, $\bar{\rho}_{X,\alpha}(u) \leqslant \bar{\rho}_{Y,\alpha}(u)$ *and* $\rho_{X,\alpha}(u) \leqslant \rho_{Y,\alpha}(u)$ *for every real* $u \in [a,b]$) *and* $\int_a^b |s|^r\, d\rho_X(s) = \int_a^b |s|^r\, d\rho_Y(s)$ *for a given* $r > \alpha-1$, *then* $F_X = F_Y$; *otherwise it implies* $\int_a^b |s|^r\, d\rho_X(s) < \int_a^b |s|^r\, d\rho_Y(s)$ *for every* $r > \alpha - 1$.

Proof. Using $P\big(X \leqslant H^{-1}(t)\big) = P\big(H(X) \leqslant t\big)$ for all $t \in [a,b]$ and applying the Fubini–Tonelli theorem, we get point (a). The proof of point (b) is practically the same as for point (b) of Proposition 2.3. ∎

The fact that a FORS measure ρ_X (associated with a FORS ordering) is a simple probability functional over a given class of random variables qualifies the FORS ordering itself. Next we propose a further characterization of FORS orderings. Suppose $|t| < +\infty$ and let

$$\rho_{X,\alpha+is}(t) = \frac{1}{\Gamma(\alpha+is)}\int_a^t (t-x)^{\alpha+is-1}d\rho_X(x)$$

be the complex extension of the FORS measure $\rho_{X,\alpha}(t)$ $(\alpha > 1)$ associated with a FORS ordering. Then, as a consequence of Lemma 2.1, for every real $\alpha > v \geqslant 1$, for all $X \in \Lambda_{(\alpha)}$ and $s,k \in R$, we get

$$\begin{aligned}
\rho_{X,\alpha+is}(t) &= \frac{1}{\Gamma(\alpha+is)}\int_{t-a}^{0} (u)^{\alpha+is-1}d\rho_X(t-u)\\
&= \frac{1}{\Gamma\big(\alpha-v+i(s-k)\big)}\int_a^t (t-u)^{\alpha-v+i(s-k)-1}\rho_{X,v+ik}(u)du.
\end{aligned}$$

Consequently, for all $v \in [1, \alpha)$ the functions

$$\Im_{X,v}(p+is) = \Gamma(p+is)\rho_{X,v+p+is}(t) = \int\limits_0^{\infty} f_v(x) x^{p+is-1} dx$$

are the Mellin transforms of the functions $f_v(x) := \rho_{X,v}(t-x) I_{[0,t-a]}(x)$, defined for all $p \in (0, \alpha - v]$ and for all $s \in R$. Thus, from the properties of the Mellin transform we get the following inversion formula: for all $v \in [1, \alpha)$, for all $X \in \Lambda_{(\alpha)}$, and for all $p \in (0, \alpha - v]$,

$$\rho_{X,v}(t-x) I_{[0,t-a]}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \Im_{X,v}(p+im) x^{-p-im} dm$$

and, in particular,

$$\rho_X(t-x) I_{[0,t-a]}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \Im_{X,1}(p+im) x^{-p-im} dm \quad \text{for all } p \in (0, \alpha - 1];$$

see, among others, Titchmarsh [36], Szmydt and Ziemian [35], and Ortobelli [24]. Observe that the Mellin transform is an analytical function. Then, if we know the values that the transform $\Im_{X,v}(s_n)$ assumes on a countable complex sequence $\{s_n\}_{n \in N}$ ($s_n \in C$) and even at its accumulation point s (i.e., $s_n \to s$), we univocally determine $\rho_{X,v}(x)$ for every $x \in [a, t]$. That is, the α fractional integral valued at a given point t and for every $\alpha \in (1, p]$ represents itself a transform because $h_X(u) = \Gamma(u-1)\rho_{X,u}(t)$ for all $u > 1$ is the Mellin transform of $\rho_X(t-x) I_{[0,t-a]}(x)$ valued on the real line. From this simple observation we get a systematic way to generate FORS orderings based on the following theorem.

THEOREM 3.1. *Suppose $|b| < +\infty$ and $\rho_X^{(1)} : [a, b] \to R$ is a simple FORS1 measure associated with a simple FORS1 ordering $\succ$ defined on a class of random variables Λ. If $\rho_X^{(1)}$ is a bounded and monotone function, then the probability functional $\rho_X^{(2)} : [1, p_1] \to R$ defined by $\rho_X^{(2)}(u) = \rho_{X,u}^{(1)}(b)$ points out a simple FORS2 measure (induced by $\succ$) on the class of random variables Λ_{p_1},*

$$\Lambda_{p_1} = \{X \in \Lambda / p_1 > 1 : |\rho_{X,p_1}^{(1)}(b)| < +\infty\}.$$

In addition, $\rho_X^{(2)}$ is associated with the following new simple FORS2 ordering induced by the previous one $\succ$ defined for every pair

$$X, Y \in \Lambda_{p_1,(\alpha)} = \Big\{Z \in \Lambda_{p_1} : \Big|\int\limits_1^{p_1} u^{\alpha-1} d\rho_X^{(2)}(u)\Big| < \infty\Big\}, \qquad \alpha > 0,$$

as follows:

$$X \underset{\succ,\alpha}{\mathrm{FORS2}}\, Y \ \textit{iff}\ \rho_{X,\alpha}^{(2)}(u) \leqslant \rho_{Y,\alpha}^{(2)}(u) \ \textit{for all}\ u \in [1, p_1].$$

We call the second level of ordering induced by $\succ$ *the new class of orderings FORS*2.

P r o o f. From the Hölder inequality we know that for every $\alpha \in (1, p]$ and $X \in \Lambda_p = \{Z \in \Lambda\colon |\rho^{(1)}_{Z,p}(b)| < +\infty\}$, it follows that $X \in \Lambda_\alpha$. Thus, the function $\rho^{(2)}_X(u) = \rho^{(1)}_{X,u}(b)$ is defined for every $X \in \Lambda_{p_1}$ and for every $u \in [1, p_1]$. Moreover, since

$$X \underset{\succ,1}{\text{FORS1}}\, Y \text{ implies } X \underset{\succ,u}{\text{FORS1}}\, Y \text{for any } u \in [1, p_1],$$

we infer that

$$X \underset{\succ,1}{\text{FORS1}}\, Y \text{ implies } \rho^{(2)}_X(u) = \rho^{(1)}_{X,u}(b) \leqslant \rho^{(1)}_{Y,u}(b) = \rho^{(2)}_Y(u).$$

Therefore $\rho^{(2)}_X(u)$ is a FORS2 measure induced by $\succ$ on the class Λ_{p_1} for any fixed $u \in [1, p_1]$. In addition, if for almost any $u \in [1, p_1]$ we have $\rho^{(2)}_X(u) = \rho^{(2)}_Y(u)$, then $\rho^{(1)}_{X,u}(b) = \rho^{(1)}_{Y,u}(b)$. Thus, by applying the inverse Mellin transform, we get $\rho^{(1)}_X(t) = \rho^{(1)}_Y(t)$ for every $t \in [a, b]$. This implies $F_X = F_Y$, i.e., $\rho^{(2)}_X$ is a simple probability functional on the class Λ_{p_1}. Thus, using Definition 3.2 we prove the theorem. ∎

Thus, given a FORS1 ordering, we can define a second level of ordering FORS2 and the definition can be extended recursively. As a matter of fact, we can easily get a k-th level of FORSk ordering $\rho^{(k)}_X\colon [1, p_k] \to R$ with $\rho^{(k)}_X(u) = \rho^{(k-1)}_{X,u}(p_{k-1})$ on the class of random variables

$$\Lambda_{p_k} = \{X \in \Lambda_{p_{k-1}} \mid p_k > 1\colon |\rho^{(k-1)}_{X,p_k}(p_{k-1})| < +\infty\},$$

where $p_0 = b$. An immediate consequence of the proposed analysis is given by the following corollary.

COROLLARY 3.2. *Under the assumption of Theorem* 3.1, *for every* $m > k$ *and* $\alpha \geqslant 1$ *the ordering* $X \underset{\succ,1}{\text{FORS}k}\, Y$ *implies* $X \underset{\succ,\alpha}{\text{FORS}m}\, Y$. *In particular*, *if* σ_X *is a FORSk probability functional induced by the k-th level of a FORS ordering* $\underset{\succ,v}{\text{FORS}k}$ $(v \geqslant 1)$, *then* σ_X *is also a FORS measure induced by order* $\succ$.

Thus, it follows from Corollary 3.2 that the new orders are finer than the generating one. This could permit us to characterize better the investors' choices under uncertainty. However, several new questions arise by the introduction of k-level orderings. For example, it could be interesting to analyze the relations/differences existing among functionals $\rho^{(k)}_{X,\alpha}$ and $\rho^{(s)}_{X,\beta}$ for $s \neq k$ and/or $\alpha, \beta > 1, \alpha \neq \beta$, in order to understand their impact on investors' preferences. We also believe that some of the "moments" properties verified by Fishburn [8] and O'Brien [22] can be extended to FORS type orderings. However, because of space constraints, we cannot be exhaustive in our analysis and further analysis of these issues will be the subject of a future paper.

3.2. Examples of FORS measures and orderings. Typical examples of FORS orderings are the classical stochastic orders and their duals that are induced by the first stochastic dominance order. Consider the following examples of FORS measures and orderings.

Moment FORS measures. For any fixed real t,

$$\rho_X(\lambda) = \Gamma(\lambda+1)F_X^{(\lambda+1)}(t) = E\big((t-X)_+^\lambda\big)$$

is a primary probability functional over the class of p-integrable random variables $\Lambda = L^p = \{X \mid E(|X|^p) < +\infty\}$. In addition, $\rho_X(\lambda)$ defined for every $\lambda \geqslant m$ and a given $m < p$ is a FORS measure induced by $\underset{m+1}{\overset{b}{>}}$. Then for every $\alpha \geqslant 1$ the measure

$$_m\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)}\int_m^u (u-s)^{\alpha-1}d\rho_X(s) \quad \text{for all } u \geqslant m$$

with $m < p$ is a FORS measure induced by $\underset{m+1}{\overset{b}{>}}$ that identifies the distribution of the tail (i.e., $_m\rho_{X,\alpha}(u) = {_m\rho_{Y,\alpha}}(u)$ for all $u \geqslant m$ for a given $\alpha \geqslant 1$ iff $F_X(x) = F_Y(x)$ for all $x \leqslant t$). This is a logical consequence of the inverse Mellin transform applied to the moment curve of the positive random variable $(t-X)_+$ that univocally determines the distribution of the tail.

Weak moment FORS orderings. Let us consider the class of random variables bounded from above and p-integrable: $\Lambda = \{Z \in L^p \mid Z \leqslant b < +\infty\}$. Then for every $m < p$ we can consider $_m\rho_X(\lambda) = E\big((b-X)^\lambda\big)$ for all $\lambda \geqslant m$, which is a FORS measure induced by the $m+1$ stochastic dominance order $\underset{m+1}{\overset{b}{>}}$, that is also a simple probability functional over the class Λ. Thus for every $m \geqslant 0$ the following probability functional:

$$_m\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)}\int_m^u (u-s)^{\alpha-1}d_m\rho_X(s) \quad \text{for all } u \geqslant m,$$

identifies a FORS ordering induced by the order $\succ \equiv \underset{m+1}{\overset{b}{>}}$. That is, for every pair of random variables X and Y in the class Λ:

$$X \underset{\succ,\alpha}{\text{FORS}}\, Y \text{ if and only if } {_m\rho_{X,\alpha}}(u) \leqslant {_m\rho_{Y,\alpha}}(u) \quad \text{for all } u \geqslant m.$$

Similar analysis can be done with random variables bounded from below. Thus for random variables $\tilde{\Lambda} = \{Z \mid -\infty < a \leqslant Z \leqslant b < +\infty\}$ bounded from below and above we can express moment FORS orderings induced by the order $\succ \equiv \underset{m+1}{>}$ assuming that for all $Z \in \tilde{\Lambda}$ we have $Z \leqslant b$ and $-Z \leqslant -a$. Thus if for all $\lambda \geqslant m$ we have ${}_m\rho_X(\lambda) = E\big((b - X)^\lambda\big)$, then ${}_m\rho_{-X}(\lambda) = E\big((X - a)^\lambda\big)$. Consequently, for every pair of random variables X and Y belonging to $\tilde{\Lambda}$, we can say that *X dominates Y in the sense of α-moment FORS uncertainty ordering induced by the risk ordering* $\succ \equiv \underset{m+1}{>}$ when $X \underset{\succ,\alpha}{\mathrm{FORS}} Y$ and $-X \underset{\succ,\alpha}{\mathrm{FORS}} -Y$.

4. CONCLUDING REMARKS

This paper unifies the classical theory of stochastic dominance and investor preferences with the recent literature on risk measures applied to the portfolio selection choice problem faced by investors. First we distinguish between primary, simple, and compound probability functionals. In addition, we propose new orderings and measures for risk and reward.

Many new problems arise from this analysis. First, since some of the "moments" properties indicated by O'Brien [22] can be extended to FORS-type orders, we can better specify the optimization portfolio problem by taking into account the investor's attitude toward risk. Therefore, if we create an ordering induced by an order of preferences, we need to propose optimization models that are based on consistent probability functionals. Second, we need to understand how to value the impact of different probability functionals. In particular, in order to determine the best opportunity from the perspective of different market agents, we have to compare the effect of several portfolio strategies. In that case, we need to consider the theoretical characteristics of the different statistics and their asymptotic behavior.

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